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Non-Hermitian Fermion Mapping for One-Component Plasma

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The two-dimensional, one-component logarithmic Coulomb gas is mapped onto a non-Hermitian fermionic field theory. At $\beta = 2$, the field theory is free. Correlation functions are calculated and a perturbation theory is discussed for extending to other β. A phase transition is found at the mean-field level at large β. Some results are extended to spaces of constant negative curvature.

KEY WORDS: One-component plasma; non-Hermitian; crystallization.

1. INTRODUCTION

The problem of the statistical mechanics of a system of particles in a two-dimensional plane, all possessing the same sign of charge and interacting via a repulsive Coulomb interaction, has been looked at by a number of authors.[1] This is referred to as a one-component plasma since all particles have the same sign of charge. The Coulomb interaction is logarithmic in two-dimensions which makes it possible to introduce a dimensionless inverse temperature β such that the partition function of an N-particle system is equal to

$$\int \left(\prod_{i=1}^{N} \right) dz_i \, d\bar{z}_i \Bigg) \prod_{i<j}^{N} |z_i - z_j|^{\beta} \prod_{i=1}^{N} e^{-\beta U(z_i)} \tag{1}$$

where $U(r)$ is some background potential and where $z = x + it$ and $\bar{z} = x - it$ are coordinates in the complex plane.

It is believed that at some β of approximately 140 there is a first order phase transition in the system.[2] Below the phase transition the system

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acquires long-range orientational order but does not acquire long-range positional order. The system is thus of interest as a model of crystallization. In the last section of the paper, we consider this model on a space of constant negative curvature. This may make it possible to look at effects of frustration since it is not possible to form a hexagonal crystal lattice without defects on a negative curvature space.[3]

This problem is exactly solvable at $\beta = 2$ for a variety of background potentials. We introduce a new method of solving the problem, based on mapping to a non-hermitian field theory, which may be easier to deal with for certain potentials. In addition, we look at a perturbation theory which permits a mean-field analysis of the phase transition in this theory.

Previously, in ref. 4, it was pointed out that an appropriate field theory could describe a two-component plasma, and that an appropriate limit of that theory would yield the one-component plasma. That limit is in fact the theory described in this paper. However, explicit calculations were performed in this theory only in the case of equal numbers of charges of both signs, and no calculation was made of the one-component plasma using the field theory technique.

In one dimension, fermion mappings onto non-hermitian free field theories exist for $\beta = 1, 2, 4$. See ref. 5.

2. FERMION MAPPING AT $\beta = 2$

As a a statistical mechanics system, the model, for a finite number of particles in a uniform background, is defined by the partition function

$$Z = \int \left(\prod_{i=1}^{N} dz_i \, d\bar{z}_i \right) \prod_{i<j}^{N} |z_i - z_j|^{\beta} \prod_{i=1}^{N} e^{-\rho |z_i|^2} \tag{2}$$

We note that the factor $\prod_{i<j}^{N} |z_i - z_j|^{\beta}$ in the partition function is proportional to a correlation function in a free bosonic field theory:

$$\left\langle \prod_{i=1}^{N} e^{i\sqrt{4\pi}\,\Phi(z_i)} e^{-iN\sqrt{4\pi}\,\Phi(\infty)} \right\rangle \tag{3}$$

where the field Φ has the action $S = (1/\beta) \int dx\, dt (\nabla\Phi)^2$ and where the factor of $e^{-iN\sqrt{4\pi}\,\Phi(\infty)}$ acts as a neutralizing background charge.

Therefore, Eq. (2) is proportional to

$$Z = \int [d\Phi] \int \left(\prod_{i=1}^{N} dz_i \, d\bar{z}_i \right) e^{-S - iN\sqrt{4\pi}\,\Phi(\infty)} \prod_{i=1}^{N} \{ e^{i\sqrt{4\pi}\,\Phi(z_i)} e^{-\rho |z_i|^2} \} \tag{4}$$

This is proportional to

$$Z = \int [d\Phi] \int e^{-S - iN\sqrt{4\pi}\,\Phi(\infty) + \int e^{-\rho|z|^2} e^{i\sqrt{4\pi}\,\Phi(z)}\,dz\,d\bar{z}} \tag{5}$$

as the perturbative expansion of the exponential in powers of $\int e^{-\rho|z|^2} \times e^{i\sqrt{4\pi}\,\Phi(z)}\,dz\,d\bar{z}$ vanishes for all except the Nth power, when we recover Eq. (4). It does not matter whether there is a prefactor in front of $\int e^{i\sqrt{4\pi}\,\Phi(x,t)}\,dx\,dt$ in Eq. (5) as this will only change the partition function by a constant factor. Similarly, in Eq. (6) below, the prefactor in front of the exponential of the field Φ is unimportant.

We may then go to an infinite number of particles, and write the partition function as

$$Z = \int [d\Phi]\, e^{-S + \int (e^{i\sqrt{4\pi}\,\Phi(x,t)} - i\rho\sqrt{4\pi}\,\Phi(x,t))\,dx\,dt} \tag{6}$$

Here, the neutralizing background charge, instead of being placed at infinity by the term $e^{-iN\sqrt{4\pi}\,\Phi(\infty)}$ is spread throughout space by the term $e^{\int -i\rho\sqrt{4\pi}\,\Phi(x,t))\,dx\,dt}$. This leads to a particle density proportional to ρ.

This quivalence between Eqs. (2) and (6) is similar to the equivalence between the statistical mechanics of an plasma consisting of both plus and minus charges and the field theory of the sine-Gordon equation.

The sine-Gordon equation at $\beta = 2$ maps onto a problem of free massive fermions, while at other temperatures the equation maps onto the Thirring model, a model of interacting massive fermions.[6] A review of this procedure, known as bosonization, may be found in the book.[7] We will follow an analogous procedure in this case, leading to a non-hermitian field theory of fermions. All correlation functions will be given at $\beta = 2$, while a perturbation theory in a four-fermion interaction will lead to other temperatures.

Each term of the bosonic action translates into a given term of an action for relativistic fermions. The action S for the field Φ translates into $2\int (\psi_R^\dagger \partial_{\bar{z}} \psi_R - \psi_L^\dagger \partial_z \psi_L)\,dx\,dt$.

The term $e^{i\sqrt{4\pi}\,\Phi(x,t)}$ translates into $2\pi a \psi_L^\dagger \psi_R$, where a is an ultraviolet cutoff for the theory. The existence of only one kind of charge in the statistical mechanics theory leads to a non-hermitian field theory.

It is seen from the previous paragraph that placing a charge in the statistical mechanics theory at a given point corresponds to turning a fermion at that point from a right-mover into a left-mover. The effect of the neutralizing background charge is to turn left-movers back into right-movers via the anomaly. One may imagine the neutralizing background charge as a charge at infinity. Unfortunately we run into problems when finding a term in the fermionic theory which corresponds to this background; we Will employ a trick of integration by parts below to circumvent these difficulties.

We can write

$$-\int i\rho \sqrt{4\pi}\, \Phi \, dx\, dt = -\int i\rho \sqrt{4\pi}\, \Phi(\partial_t t)\, dx\, dt \tag{7}$$

and integrate by parts with respect to t to turn this into

$$\int i\rho \sqrt{4\pi}\, t(\partial_t \Phi)\, dx\, dt + \{\text{boundary terms}\} \tag{8}$$

This then becomes in the fermionic theory equal to

$$\int \rho t 2\pi J_x \, dx\, dt + \{\text{boundary terms}\} \tag{9}$$

where $J_x = \psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R$. This has the effect of introducing a time-dependent chemical potential which has different signs for the right- and left-movers. If we imagine following the Hamiltonian evolution of the fermionic field theory in imaginary time, right-moving particles are constantly converted into left moving particles, but the energy of the states keeps changing so that on average the number of holes in the negative energy sea is the same for the both chiralities. The effect of the boundary term resulting from the integration by parts is that we must start at $t = -\infty$ with a state in which all right-moving states (of both positive and negative energy) are filled and all left-moving states are empty and end at $t = \infty$ with a state in which all left-moving states are filled and all right-moving states are empty. Let us refer to the state at $t = -\infty$ as the state V^-. We will refer to the state at $t = +\infty$ as the state V^+.

The final action for the fermion field is

$$\int \{2(\psi_R^\dagger \partial_{\bar{z}} \psi_R - \psi_L^\dagger \partial_z \psi_L) + \sqrt{2\rho}\, \psi_L^\dagger \psi_R + \rho t 2\pi(\psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R)\}\, dx\, dt \tag{10}$$

The factor $\sqrt{2\rho}$ is chosen to cancel a factor arising later from an integral of $\int e^{-2\pi\rho t^2}\, dt$. It also serves to give the term in the action the correct dimensions. This factor is unimportant since the fixed density of the background charge means, due to charge neutrality, that the number of particles in the statistical mechanics problem is fixed and thus this factor just multiplies the partition function but does not affect the physics. This is the analogue of the statement above, that prefactors in front the exponentials of the Φ field in Eqs. (5) and (6) were unimportant. It is simply a matter of convenience to choose this term to be $\sqrt{2p}$ instead of $2\pi a$ as would be expected from the bosonization procedure.

In Eq. (10), there is freedom to choose different representations of the background charge, which correspond to different gauges. The choice we have made above is the choice which makes the calculation of correlation functions the simplest. Another possiblity would be to replace tJ_x by

$$\frac{tJ_x - xJ_t}{2} \tag{11}$$

where $J_t = i(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L)$. This gauge has the advantage of leading to rotationally and translationally invariant correlation functions.

The freedom to choose different gauges is analogous to the freedom in magnetic problems to choose different gauges. The gauge in Eq. (10) is analogous to Landau gauge. The gauge in Eq. (11) is analogous to symmetric gauge.

3. CORRELATION FUNCTIONS AT $\beta = 2$

3.1. Introduction

We compute correlation functions for the fermionic field and use them to then obtain correlation functions in the statistical mechanics problem. It will hopefully be clear in which theory a correlation function is being calculated.

We will compute propagators for the fermion field as follows: we will first compute the propagators for the case in which the two operators lie on the line $t = 0$, using the gauge of Eq. (10). Then we will, in the gauge of Eq. (11) discussed above, generalize to arbitrary position of the two operators.

Instead of using a functional integral formalism to compute correlation functions, we will use the operator formalism. In the operator formalism, we can write a two-point correlation function of fields $\psi(1)$, $\psi(2)$, when both points 1 and 2 are on the line $t = 0$, as the expectation value

$$\frac{1}{Z} \langle V^+ |\, \mathrm{T} e^{-\int_0^\infty H(t)\, dt} \psi(1)\, \psi(2)\, e^{-\int_{-\infty}^0 H(t)\, dt}\, | V^- \rangle \tag{12}$$

where

$$Z = \langle V^+ |\, \mathrm{T} e^{-\int_0^\infty H(t)\, dt} e^{-\int_{-\infty}^0 H(t)\, dt}\, | V^- \rangle \tag{13}$$

In these equations, $H(t)$ is the Hamiltonian of the field theory at time t. The time-ordered exponential of the integral of this Hamiltonian serves to propagate the vacuum from infinite time to time $t=0$.

The term in the Hamiltonian, $\int \psi_L^\dagger \psi_R \, dx \, dt$, can be written in Fourier components as $\int (dk/2\pi) \, a^\dagger(k)_L \, a(-k)_R$. Each term in this integral is an eigenoperator of the rest of the Hamiltonian, with eigenvalue $2k + 4\pi\rho t$, and so one may rewrite Eq. (12) in terms of operators acting just at time $t=0$ as follows

$$\frac{1}{Z} \langle V^+ | \, e^{-\int (dk/2\pi)(\int_0^\infty e^{-2\pi\rho t^2} + 2kt) \, dt) \sqrt{2\rho} \, a^\dagger(k)_L \, a(-k)_R} \psi(1) \, \psi(2)$$

$$\times e^{-\int (dk/2\pi)(\int_{-\infty}^0 e^{-2\pi\rho t^2} + 2kt) \, dt) \sqrt{2\rho} \, a^\dagger(k)_L \, a(-k)_R} \, | V^- \rangle \tag{14}$$

3.2. $\langle a_L^\dagger a_R \rangle$

The simplest two-point function to compute is $\langle a^\dagger(k)_L \, a(-k)_R \rangle$. In the expectation value of Eq. (14), for each k the operators $a^\dagger(k)_L$ and $a(k)_R$ must appear exactly once. It follows that $Z = \prod_k e^{\int_{-\infty}^{\infty} \sqrt{2\rho} \, e^{2\pi\rho t^2 + 2kt} \, dt}$. This is

$$\prod_k e^{k^2/2\pi\rho} \tag{15}$$

By inserting the operator $a^\dagger(k)_L \, a(-k)_R$ into the expectation value, we remove one term from the product of Eq. (15). This term is $e^{k^2/2\pi\rho}$. Therefore, $\langle a^\dagger(k)_L \, a(-k)_R \rangle = e^{-(k^2/2\pi\rho)}$. In real space this is $\sqrt{\rho/2} \, e^{-(\pi/2)\rho x^2}$, and in the gauge of Eq. (11) we can generalize to arbitrary positions of the two operators so that it becomes $\sqrt{\rho/2} \, e^{-(\pi/2)\rho |z|^2}$, where z is the spacing between operators.

3.3. $\langle a_L^\dagger a_L \rangle$, $\langle a_R^\dagger a_R \rangle$

We next compute the function $\langle a^\dagger(k)_L \, a(-k)_L \rangle$. We note that initially all left-moving states are unoccupied at $t = -\infty$. Then all states are occupied at $t = \infty$. If the state with momentum k is occupied at $t=0$ then this gives a contribution of 1 to the desired two-point function. If it is unoccupied, then there is a contribution of 0. Therefore the desired two-point function is

$$\frac{\int_{-\infty}^{0} e^{-2\pi\rho t^2 + 2kt} \, dt}{\int_{-\infty}^{\infty} e^{-2\pi\rho t^2 + 2kt} \, dt} \tag{16}$$

In real space, this is $1/(2\pi i\bar{z})\, e^{-\pi(\rho/2)\,|z|^2}$, where we use the gauge of Eq. (11) to extend correlation functions off the line $t=0$.

The function $\langle a^\dagger(k)_R\, a(-k)_R\rangle$, in which both left-moving operators are replaced by right-moving operators, is similar. One finds $\langle a^\dagger(z)_R\, a(0)_R\rangle = (1/2\pi i z)\, e^{-\pi(\rho/2)\,|z|^2}$.

3.4. $\langle a_R^\dagger a_L\rangle$

The most interesting two-point function to compute is $\langle a^\dagger(k)_R\, a(-k)_L\rangle$. This requires that one $a^\dagger(k)_L\, a(-k)_R$ operator appear at time $t<0$ and one must appear at time $t>0$. Therefore, the two-point function is

$$\sqrt{2\rho}\,\frac{\int_{-\infty}^{0} e^{-2\pi\rho t^2+2kt}\,dt \int_0^\infty e^{-2\pi\rho t'^2+2kt'}\,dt'}{\int_{-\infty}^{\infty} e^{-2\pi\rho t^2+2kt}\,dt} \tag{17}$$

In real space, this is $2\rho\int_{-\infty}^{0} e^{-2\pi\rho t'^2}\,dt' \int_0^\infty e^{-2\pi\rho t^2}\,dt \int (dk/2\pi)\, e^{-k^2/2\pi\rho} \times e^{k(ix+2t+2t')}$. That becomes

$$\rho\int_{-\infty}^{0} e^{-2\pi\rho t'^2}\,dt' \int_0^\infty e^{-2\pi\rho t^2}\,dt\, e^{(\pi\rho/2)(ix+2t+2t')^2}\sqrt{2\rho} \tag{18}$$

This is equal to $\sqrt{2}\,\rho^{3/2}\int_{-\infty}^{0} dt' \int_0^\infty dt\, e^{(\pi\rho/2)(-x^2+8tt'+4ix(t+t'))}$. Integrate t' to obtain $(1/2\pi)\sqrt{\rho/2}\int_0^\infty dt((1/(t+(ix/2)))\, e^{(\pi\rho/2)(-x^2+ix4t)}$. This is equal to $(1/2\pi)\sqrt{\rho/2}\, e^{+(\pi\rho/2)\,x^2}\int_{\pi\rho x^2}^{\infty} e^{-t}/t$ which becomes

$$\frac{1}{2\pi}\sqrt{\frac{\rho}{2}}\, e^{+(\pi\rho/2)\,|z|^2}\,\mathrm{Ei}(\pi\rho\,|z|^2) \tag{19}$$

where Ei is the exponential integral function where again we use the gauge of Eq. (11) to generalize to arbitrary position of operators.

3.5. Summary of Propagators

These four propagators can be written as one propagator using a matrix notation. One finds that

$$\langle\psi^\dagger(z)_{L,R}\,\psi(0)_{L,R}\rangle = \begin{pmatrix} \dfrac{1}{2\pi i\bar{z}}\, e^{-\pi(\rho/2)\,|z|^2} & \sqrt{\dfrac{\rho}{2}}\, e^{-(\pi/2)\,\rho\,|z|^2} \\ \dfrac{1}{2\pi}\sqrt{\dfrac{\rho}{2}}\, e^{+(\pi\rho/2)\,|z|^2}\,\mathrm{Ei}(\pi\rho\,|z|^2) & \dfrac{1}{2\pi i z}\, e^{-\pi(\rho/2)\,|z|^2} \end{pmatrix} \tag{20}$$

where the top row of the matrix denotes the left chirality for the creation operator and the bottom row denotes the right chirality for the creation operator. Similarity the left row of the matrix denotes the left chirality for the destruction operator and the right row denotes the right chirality for the destruction operator.

3.6. Statistical Mechanics

Having computed the propagators, it is possible to compute correlation functions in the statistical mechanics theory. Let us compute the two-point particle correlation function at $\beta = 2$. This is obtained by the expectation value $\langle \sqrt{2\rho}\, \psi_L^\dagger(0)\, \psi_R(0)\, \sqrt{2\rho}\, \psi_L^\dagger(z)\, \psi_R(z) \rangle$. There are two ways to pair off the operators in this expression using Wick's theorem, either pairing the operators at the same point or at different points. The result is

$$\rho^2(1 - e^{-\pi\rho\, |z|^2}) \tag{21}$$

which agrees with the known result.

4. PERTURBATION THEORY

By adding to the action of the fermionic theory a term of the form $(\bar{\psi}\psi)^2$, we will, due to the bosonization rules, change the value of β. An attractive interaction will raise β, while a repulsive interaction will lower β. We will employ a simple mean-field type procedure: the interaction will be decoupled into an interaction with an external gauge field. Then, we will integrate out the fermionic field to obtain an action for the gauge field. It will be found that at sufficiently high, finite β the quadratic term in the action for the gauge field will change sign. This will be identified as the location of a second order phase transition. It will then be argued that cubic terms appear in the action for the gauge field, and will convert the phase transition to first order.

The approximation involved in the perturbation theory is purely a mathematical approximation. The perturbation theory should give correct results order by order in the expansion parameter, which is $(1/2) - (1/\beta)$, but the extension below of the one loop calculation to large values of the parameter (that is, to the phase transition point) should not be trusted that accurately.

In Eq. (6), we may write the action S as $(1/2) \int dx\, dt (\nabla\Phi)^2 + (1/\beta - 1/2) \int dx\, dt (\nabla\Phi)^2$. The first term can be bosonized as before. The second term is equal to $(1/\beta - 1/2)(:\psi_R^\dagger \psi_R:)(:\psi_L^\dagger \psi_L:)$. We define the normal ordering of the fermion fields simply be requiring that, in the perturbative diagram expansion, using the gauge of Eq. (11), one does not include tadpole diagrams, in which two operators are contracted at the same point. This in fact agrees with the usual definition of normal ordering, that one define $:\psi_R^\dagger \psi_R:$ to have vanishing expectation value in the vacuum of the massless, non-interacting fermion system.

For a given value of β, if we choose to use the gauge of Eq. (10), the desired fermionic action is

$$\int \left\{ 2(\psi_R^\dagger \partial_{\bar{z}} \psi_R - \psi_L^\dagger \partial_z \psi_L) + \sqrt{2\rho}\, \psi_L^\dagger \psi_R + \rho t 2\pi (\psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R) \right.$$
$$\left. + \left(\frac{1}{\beta} - \frac{1}{2}\right) 4\pi (:\psi_R^\dagger \psi_R:)(:\psi_L^\dagger \psi_L:) \right\} dx\, dt \tag{22}$$

For calculations below, we use the gauge of Eq. (11).

We can write the interaction term by introducing an external gauge field $A_{R,L}$. The action for the gauge field plus the coupling between the gauge field and the fermion field is

$$\int \left\{ A_R J_R + A_L J_L + \frac{1}{4\pi} \frac{1}{(1/2) - (1/\beta)} A_R A_L \right\} dx\, dt \tag{23}$$

where we define $J_R = :\psi_R^\dagger \psi_R:$ and $J_L = :\psi_L^\dagger \psi_L:$ and where A_R is the complex conjugate of A_L.

It will turn out, when we compute various diagrams, that results will not be gauge invariant. This is unrelated with the particle choice of gauge we made to compute propagators, and instead is a result of the boundary conditions due to the neutralizing background field. The theory is also not invariant under charge conjugation. This implies that Furry's theorem[8] need not hold, and we may obtain contributions from fermion diagrams with an odd number of photon vertices attached. This will then convert the second order transition into a weakly first order transition.

Let us first compute the quadratic terms in the action for $A_{R,L}$ resulting from integrating out ψ. There are three different types of diagrams which must be considered, which involve computing different current-current correlation functions. We may have to compute a correlation function involving two currents of the same chirality. This will be considered first. We may have to compute a correlation function involving currents of the opposite chirality. This correlation function splits into two pieces: one piece which may be obtained by naively calculating diagrams, and one piece which is a result of the need for a regulator field to get rid of divergences.

The introduction of the quadratic term will change the action of the gauge field, as a function of wavevector k, to

$$\frac{1}{4\pi}\frac{1}{(1/2)-(1/\beta)}A_R(k)\,A_L(-k)-\frac{1}{2}\langle J_R(k)\,J_R(-k)\rangle\,A_R(k)\,A_R(-k)$$
$$-\frac{1}{2}\langle J_L(k)\,J_L(-k)\rangle\,A_L(k)\,A_L(-k)-\langle J_R(k)\,J_L(-k)\rangle\,A_R(k)\,A_L(-k)) \tag{24}$$

All that needs to be computed now are some current-current correlators.

The current-current correlation function $\langle J_R(0)\,J_R(z)\rangle$ is equal to $(1/(2\pi z)^2)\,e^{-\pi\rho\,|z|^2}$. We must take the Fourier transform of this at a given momentum k. In order to do this, it is necessary to introduce a massive regulator field. Without the term $e^{-\pi\rho\,|z|^2}$ in the correlator, the Fourier transform of $(1/(2\pi z)^2)$ is known to be $(1/4\pi)(k_L^2/|k|^2)$, where k_L is equal to $k_x - ik_t$, k_R is equal to $k_x + ik_t$, and $|k|^2 = k_L k_R$. The term $e^{-\pi\rho\,|z|^2}$ multiplies the rest of the correlator in real space and thus convolves with the correlator in Fourier space. The kth Fourier component of the current-current correlator in momentum space is then

$$\langle J_R(k)\,J_R(-k)\rangle = \int dl_L\,dl_R\,\frac{1}{4\pi}\frac{l_L}{l_R}\,e^{-(|k-l|^2/4\pi\rho)} \tag{25}$$

This integral can be performed analytically and is equal to

$$\langle J_R(k)\,J_R(-k)\rangle = \frac{1}{4\pi}\frac{k_L^2}{|k|^2}\,(1-e^{-(|k|^2/4\pi\rho)}) \tag{26}$$

It is seen that at large k the current-current correlator is unchanged from the correlator for a fermionic system with no non-hermitian term. The correlation function of two left-moving currents is calculated in the same way, simply replacing k_L by k_R and vice-versa.

The naive contribution to the correlator of two currents of opposite chirality is given by $\langle J_R(0)\,J_L(z)\rangle$. This is $(\rho/4\pi)\,\mathrm{Ei}(\pi\rho\,|z|^2)$. The Fourier transform is $(\rho/4\pi)\int dx\,dt\int_1^\infty (da/a)\,e^{ikx-\pi\rho a(x^2+t^2)}$. Doing the integrals over x and t first, we obtain

$$\frac{1}{4\pi}\int_1^\infty \frac{da}{a^2}\,e^{-(|k|^2/4a\pi\rho)} = \frac{\rho}{k^2}\,(1-e^{-k^2/4\pi\rho}) \tag{27}$$

The necessity of introducing a massive regulator field to compute the correlation function of two currents of the same chirality gives an additional contribution to the correlation function of two currents of opposite chirality. This piece is infinitely short range, and so independent of k. It is equal to $(1/4\pi)$. For the fermionic system with no non-hermitian term, such a piece is needed to maintain gauge invariance.

Adding the two contributions to the correlation function of two currents of opposite chirality yields

$$\langle J_L(k)\, J_R(-k)\rangle = \frac{1}{4\pi} + \frac{\rho}{k^2}\left(1 - e^{-k^2/4\pi\rho}\right) \tag{28}$$

Even without looking closely at the Fourier transform of the current-current correlators, we may see that there will be an instability for finite β at the level of the quadratic action for $A_{L,R}$. For the fermionic system with ho non-hermitian term, the quadratic term changes sign, for *all* k transverse to the direction of the gauge field, at $\beta = \infty$, which corresponds to a finite value of the attractive interaction. By adding the non-hermitian terms to the action, we have altered the quadratic terms. The correlation function of two currents of the same chirality has been reduced, but the reduction decays exponentially for large $|k|^2$. The correlation function of two currents of the opposite chirality has been increased by an amount which decays only algebraically for large $|k|^2$. Thus, for sufficiently large $|k|^2$, the term which is added to the action of the $A_{L,R}$ field is increased and the intability will occur at a lower value of the attractive interaction. This corresponds to an instability at a finite value of β.

Let us locate the transition temperature in this lowest order theory. The terms in the action given by Eq. (24) due to the current-current correlators are greatest when the field A is transverse. The sum of these terms then is (summing all different terms and taking $A_R = A_L$)

$$\frac{1}{4\pi} + \frac{1}{4\pi}\left(1 - e^{-k^2/4\pi\rho}\right) + \frac{1}{k^2}\left(1 - e^{-k^2/4\pi\rho}\right) \tag{29}$$

This function is equal to $(1/2\pi)$ at $k = 0$ and $k = \infty$. The function increases in magnitude as k increases from 0, until it hits a maximum, then decreases again. Numerically, the maximum is at $k = 4.74711\,\sqrt{\rho}$. At this wavevector, we find that the theory goes unstable at a temperature of $\beta = 15.4036$; here, the theory going unstable means that at this wavevector, the action given by Eq. (24) changes sign. At other wavevectors, and for the case when the gauge field is not transverse, one will find that the action changes sign at a larger value of β.

In a mean-field approximation, in the absence of cubic terms, the instability discussed in the quadratic action for $A_{R,L}$, would lead to a second order transition. It will now be shown that cubic terms are non-vanishing and that this transition becomes first order. Due to the complexity of the diagrams, we will not explicitly compute the cubic and quartic terms, and will simply show that the cubic term is non-vanishing.

In 2 dimensions Furry's theorem may be proved for a massive or massless Dirac field. If we have a fermion loop with an odd number of vertices, we may imagine reversing the direction of the propagation of the fermion around the loop. This will change the sign of all propagators which preserve chirality (propagators of the form $\langle \psi_R^\dagger \psi_R \rangle$ or $\langle \psi_L^\dagger \psi_L \rangle$) of the fermion field, and preserve the sign of all propagators that change chirality. Since the fermion must have the same chirality after it goes around the loop, there are an even number of propagators which change chirality, and thus an odd number of propagators which preserve chirality. This means that the total sign of the diagram changes when reversing the direction of propagation around the loop and thus the total contribution from both directions is zero.

For the non-hermitian theory considered here, the above proof breaks down. When reversing the direction of propagation around the loop, we may change some propagators from $\langle \psi_L^\dagger(z_1)\, \psi_R(z_2) \rangle$ to $\langle \psi_R^\dagger(z_2)\, \psi_L(z_1) \rangle$. Since the non-hermitian theory lacks charge conjugation invariance, these two propagators have different values, and thus the two contributions do not cancel. This means that cubic terms do appear in the action for $A_{R,L}$ and the transition becomes first order. It should be noted that these cubic terms are intrinsically not gauge invariant, since they vanish in a gauge in which $A_L = 0$ or $A_R = 0$.

5. EXTENSION TO CURVED SPACE

We will also consider this model on a space of constant negative curvature. We may define such a space in the half-plane given by $Im(z) > 0$, or equivalently, $t > 0$. We will use a conformal gauge for the metric such that $ds^2 = (2/Rt^2)(dx^2 + dt^2)$. Then the scalar curvature is equal to R.

It is possible to transcribe the action for the bosonic field to this curved space. One obtains

$$Z = \int [d\Phi]\; e^{-S + (2/Rt^2) \int \{e^{i\sqrt{4\pi}\,\Phi(x,t)} - i\rho\sqrt{4\pi}\,\Phi(x,t)\}\, dx\, dt} \tag{30}$$

The factor of $(2/Rt^2)$ is due to the volume element in the curved space.

There is one point in this theory which requires caution. In the flat space problem, given by Eq. (2), one must introduce a length scale a to properly define the partition function. Thus, the term $|z_i - z_j|^\beta$ in Eq. (2)

should be written $|(z_i - z_j)/a|^\beta$; the length scale is also needed to properly define particle-background and background-background interaction. In the bosonic field theory, the ultraviolet cutoff provides this length scale. However, one would like to define the length scale to be constant when measured in the metric in the negative curvature space; this means that in order to maintain the same ultraviolet cutoff a when measured in length $\sqrt{ds^2}$ everywhere, the ultraviolet cutoff when measured in length $\sqrt{dx^2 + dt^2}$ must vary proportional to t. Therefore, the operator $\int e^{i\sqrt{4\pi}\,\Phi(x,t)}$ bosonizes into something proportional to $(2\pi t)\,\psi_L^\dagger \psi_R$. Because the ultraviolet cutoff is now position dependent, one must now be careful about the ultraviolet cutoff when finding the fermion action.

One finds that the fermionic action for the curved space problem is

$$\int \left\{ 2(\psi_R^\dagger \partial_{\bar{z}} \psi_R - \psi_L^\dagger \partial_z \psi_L) + \frac{1}{t}\psi_L^\dagger \psi_R + 4\pi \frac{\rho}{Rt}(\psi_R^\dagger \psi_R - \psi_L^\dagger \psi_L) \right\} dx\, dt \tag{31}$$

If we write the partition function resulting from this action in terms of particle positions in the upper half plane, we obtain

$$\int \prod_{i=1}^{N} \frac{dz_i\, d\bar{z}_i}{t_i} \prod_{i<j} |z_i - z_j|^\beta \prod_i e^{-\beta U(z_i)} \tag{32}$$

where U is the potential arising from a background charge equal to $(2\rho/Rt^2)$; thus, $U = -4\pi(\rho/R)\ln(t)$.

In action of Eq. (31), the ratio ρ/R provides a measure of the particle density in terms of the curvature of the manifold. As this ratio tends to infinity, we expect to recover the flat space results.

In Eq. (31) there are two places in which factors of $1/t$ occur. These arise for different reasons. The factor in front of the $\psi_L^\dagger \psi_R$ arises from a combination of the $1/t^2$ of the volume element and the factor of t arising from the ultraviolet cutoff as discussed above; this is also the reason for the factor of $1/t$ in Eq. (32). The other factor arises from integrating the $-\int (2i/Rt^2)\,\rho\,\sqrt{4\pi}\,\Phi(x,t)\,dt$ term by parts to turn it into $\int (2i/Rt)\,\rho \times \partial_t(\sqrt{4\pi}\,\Phi(x,t))\,dt$.

It is now possible to proceed as before and calculate propagators. We will simplify and sketch the calculation of only $\langle \psi_L^\dagger \psi_R \rangle$. We restrict to the case in which both operators are on the line $t = 1$. The only new feature that emerges compared to the previous calculation in flat space is that the final state is different. The initial state, $|V^-\rangle$, is still a state in which all right-moving states are occupied and all left-moving states are empty. The system starts in this state at time $t = 0$. However, there is no term left from the integration by parts at $t = \infty$ and therefore the system ends in the vacuum state $|V\rangle$ which is the normal vacuum state for a the free fermion theory, where all negative energy states are filled and all positive energy states are empty.

Following a similar procedure as before, we move all $\psi_L^\dagger \psi_R$ operators resulting from the Hamiltonian evolution to the time $t=1$. There, an operator $a^\dagger(k)_L\, a(-k)_R$ comes with the amplitude

$$\int_0^\infty dt\, \frac{1}{t}\, e^{-\int_t^1 2k + 8\pi\rho/(Rt')\, dt'} \tag{33}$$

This is equal to $\int_0^\infty dt(1/t^{1-8\pi\rho/R})\, e^{2k(t-1)}$, which is proportional to $e^{-2k}(1/|k|^{8\pi\rho/R})$. Therefore, $\langle a^\dagger(k)_L\, a(-k)_R\rangle$ at $t=0$ is proportional to $e^{2k}\,|k|^{8\pi\rho/R}$. Due to the initial and final states, $|V^-\rangle$ and $|V\rangle$, the propagator is non-vanishing only for $k<0$, when this Fourier transforms to $1/(1-i(x/2))^{1+8\pi\rho/R}$.

The result for the correlation function in the statistical mechanics problem, obtained from calculating $\langle \psi^\dagger(0)_L\, \psi(0)_R\, \psi(x)_L\, \psi(x)_R\rangle$, is

$$1-\left(\frac{1}{1+(x^2/4)}\right)^{1+8\pi\rho/R} \tag{34}$$

Recall that the distance between two points on the line $t=1$, located at $x=0$ and $x=x$, is, measured in the metric for the negative curvature space, less than x. In reality it goes logarithmically with x for large x. If the distance between the two points in the curved space metric is equal to s, then it may be shown that

$$x = 2\,\mathrm{Sinh}\left(\sqrt{\frac{R}{2}}\,\frac{s}{2}\right) \tag{35}$$

We find that the correlation function of two charges in the statistical mechanics problem is

$$1-\left(\frac{1}{1+\mathrm{Sinh}(\sqrt{R/2}(s/2))^2}\right)^{1+8\pi\rho/R} \tag{36}$$

Thus, the actual decay of correlations in the curved space problem is exponential, as would be expected from a high-temperature expansion. The slower decay of the correlations in the curved space case, exponential instead of Gaussian, indicates that screening is less effective than in flat space. If we take a limit as ρ/R goes to infinity, the correlation function turns back into a Gaussian, as in the flat space case.

6. CONCLUSION

A new formulation has been given of the one-component plasma problem. This provides an alternative way of deriving old results. This technique may turn out to be simpler than others for certain background potentials. In addition, the perturbation theory of this model in terms of fermionic operators is very different from the previously developed perturbation theory for the statistical mechanics theory,[9] and may lead to easier calculations.

In particular, we located an instability of the theory at the quadratic level. It must be noted that the location of the second order transition in this theory, at $\beta = 15.4036$, is approximately an order of magnitude lower than the actual location of a first order transition. Also, the appearance of cubic terms, while converting the second order transition to first order, will further lower the transition β. This is evidence that the higher order corrections to the effective action for the gauge field are not negligible. However, the manner in which the transition temperature is calculated makes this number very sensitive to small adjustments in the effective action; one must calculate the response functions in the fermionic theory and then relate β to the reciprocal of the change in the action for the gauge theory. The act of taking the reciprocal makes this procedure less accurate. Unfortunately, we are also unable to understand the particular wavevector which goes unstable, as we do not see how to easily relate this wavevector to any spacing in a triangular lattice. In addition, since the order parameter is a vector, it is possible to construct a mean-field state that breaks orientational symmetry but not translational symmetry (constant non-vanishing gauge field). This would require that the first wavevector to go unstable would be at $k = 0$. This does not happen yet at the one-loop level, though one would expect it would happen to higher orders.

However, we may hope that the perturbation series for the effective action of the gauge field will be convergent, order by order in $(1/2 - 1/\beta)$, as the only actual instability of the theory is at $\beta = \infty$. This implies a radius of convergence of 1/2, and thus the theory should converge for $1 < \beta < \infty$. Arguments like Dyson's instability argument for the non-convergence of the perturbation series do not apply here since there is no instability near $\beta = 2$.[10] To make this convergence more clear, it may help to adjust units so that the action for the gauge field is

$$\int \left\{ A_R A_L + \sqrt{4\pi \left(\frac{1}{2} - \frac{1}{\beta} \right)} \, (A_R J_R + A_L J_L) \right\} dx \, dt \tag{37}$$

to avoid what may look like a singularity at $\beta = 2$. Thus, a sufficiently high order calculation in this theory should yield the effective action for the gauge field, which can then be treated in a mean-field fashion to, in principle, extract the transition properties with arbitrary accuracy. Of course, the series might diverge at the point of a second order phase transition due to infrared problems, but since a first order phase transition is expected to occur before the second order transition, this is not a problem.

Finally, the model has been considered on a curved manifold. The two-point correlations in the original statistical model have an exponential decay instead of a Gaussian decay. Since the decay of this quantity reflects the effects of screening, it seems that on a curved manifold the system screens less well. This may be flue to an effect of frustration, introduced by the curvature. It would be interesting to extend the perturbation theory to a curved manifold, both to look at the RPA as well as to look at correlation functions away from $\beta = 2$. It is believed from a perturbation theory for the original statistical mechanics model[9] that the correlation functions on a flat space begin to show short-range order as soon as $\beta > 2$. This means a lowest order perturbation theory calculation for the curved space may show interesting frustration effects.

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REFERENCES

1. A. Alastuey and B. Jancovici, *J. Phys. (France)* **42**:1 (1981); J. Ginibre, *J. Math. Phys.* **6**:440 (1985); F. Cornu, B. Jancovici, and L. Blum, *J. Stat. Phys.* **50**:1221 (1988).
2. J. M. Caillol *et al., J. Stat. Phys.* **28**:325, (1982).
3. D. R. Nelson, *Phys. Reb. B* **28**:5515 (1983); D. R. Nelson, in *Topological Disorder in Condensed Matter*, F. Yonezawa and T. Ninomiya, eds. (Springer, Berlin, 1983).
4. F. Cornu and B. Jancovici, *Europhys. Lett.* **5**:125 (1988).
5. M. B. Hastings, cond-mat/9612122.
6. S. Coleman, *Phys. Rev. D* **11**:2088 (1975).
7. A. M. Tsvelik, *Quantum Field Theory in Condensed Matter Physics* (Cambridge University Press, Cambridge, 1996).
8. C. Itzykson and J. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
9. B. Jancovici, *Phys. Rev. Lett.* **46**:386 (1981).
10. F. J. Dyson, *Phys. Rev.* **85**:631 (1952).